Varianta 038

Subiectul I

a) $-1+i$. b) -1 si 1. c) 0. d) 0. e) $c=2; d=1$. f) $\mathrm{M}(1;2)$.

Subiectul II

1. a) $b-a=1$. b) 0. c) $\det(A)=\det(B)=-2$. d) 0. e) $\{1.2.3\}$.
2. a) 0.b) $\ln 2$. c) $x=1$. d) $f''(x)>0, \forall x\in \mathbf{R}$. e) $f(x)=x$.

Subiectul III

$a)\ x=a; y=b.$

$b)\ \det(U)=-1, rangU=2.$

$c)\ U^2=I_2, U^3=U.$

$d)\ U^{2007}=\left(U^2\right)^{1003}\cdot U=I_2^{1003}\cdot U=U.$

e) Deoarece $(aI_2)\cdot(bU)=(bU)\cdot(aI_2)$ avem.

$A^n=C_n^0a^nI_2+C_n^1a^{n-1}bU+C_n^2a^{n-2}b^2U^2+...+C_n^nb^nU^n, (\forall)n\in \mathbf{N}^*.$

Avem $U^{2k}=I_2$ si $U^{2k+1}=U$.

Asadar :

$A^n=(C_n^0a^n+C_n^2a^{n-2}b^2+...)\cdot I_2+(C_n^1a^{n-i}b+C_n^3a^{n-3}b^3+...)\cdot U=$

$=\dfrac{(a+b)^n+(a-b)^n}{2}I_2+\dfrac{(a+b)^n+(a-b)^n}{2}U, (\forall)n\in \mathbf{N}^*.$

f) Fie $X=\begin{pmatrix} x & y\\ z & t\end{pmatrix}\in \mathrm{M}_2(\mathbf{R})$.

Din $\begin{pmatrix} 1 & 2\\ 2 & 1\end{pmatrix}X=X\begin{pmatrix} 1 & 2\\ 2 & 1\end{pmatrix}, \Rightarrow \begin{pmatrix} x+2z & y+2t\\ 2x+z & 2y+t\end{pmatrix}=\begin{pmatrix} x+2y & 2x+t\\ z+2t & 2z+t\end{pmatrix}$.

Asadar $x=t$ si $y=z$. Deci $\Rightarrow u,v\in \mathbf{R}$ astfel ca $X=\begin{pmatrix} u & v\\ v & u\end{pmatrix}$.

g) Deoarece $X\begin{pmatrix} 1 & 2\\ 2 & 1\end{pmatrix}=X\cdot X^{2007}=X^{2007}\cdot X=\begin{pmatrix} 1 & 2\\ 2 & 1\end{pmatrix}X$ din f) $\Rightarrow \exists\, u,v\in \mathbf{R}$ astfel ca $X=\begin{pmatrix} u & v\\ v & u\end{pmatrix}$.

Din e) $X^{2007}=\begin{pmatrix} \dfrac{(u+v)^{2007}+(u-v)^{2007}}{2} & \dfrac{(u+v)^{2007}-(u-v)^{2007}}{2}\\ \dfrac{(u+v)^{2007}-(u-v)^{2007}}{2} & \dfrac{(u+v)^{2007}+(u-v)^{2007}}{2}\end{pmatrix}$.

Obtinem sistemul : $\begin{cases}(u+v)^{2007}+(u-v)^{2007}=2\\ (u+v)^{2007}-(u-v)^{2007}=4\end{cases} \Rightarrow \begin{cases}(u+v)^{2007}=3\\ (u-v)^{2007}=-1.\end{cases}$

Deci $\begin{cases} u+v=\sqrt[2007]{3} \\ u-v=-1. \end{cases}$

Deducem ca solutia ecuatiei este $X=\begin{pmatrix} \frac{\sqrt[2007]{3}-1}{2} & \frac{\sqrt[2007]{3}+1}{2} \\ \frac{\sqrt[2007]{3}+1}{2} & \frac{\sqrt[2007]{3}-1}{2} \end{pmatrix}$.

Subiectul IV

a) $a_0=\int_0^{\pi}\left(\frac{1}{2\pi}x^2-x\right)dx=\frac{\pi^2}{6}-\frac{\pi^2}{2}=-\frac{\pi^2}{3}$.

b) Se utilizeaza formula de integrare prin parti.

c) Conform punctului b) avem:

$b_n=a_1+a_2+...+a_n=\sum_{k=1}^{n}a_k=\int_0^{\pi}\left(\frac{1}{2\pi}x^2-x\right)\sum_{k=1}^{n}\cos kxdx, (\forall)n\in\mathbf{N}^*;$

d) Avem $2\sin\frac{x}{2}\cos x=\sin\frac{3x}{2}-\sin\frac{x}{2}$; $2\sin\frac{x}{2}\cos 2x=\sin\frac{5x}{2}-\sin\frac{3x}{2}$.....

$2\sin\frac{x}{2}\cos nx=\sin\left(\frac{x}{2}+nx\right)-\sin\left(nx-\frac{x}{2}\right)$.

Deducem ca

$\left(2\sin\frac{x}{2}\right)\sum_{k=1}^{n}\cos kx=\sin\left(nx+\frac{x}{2}\right)-\sin\left(\frac{x}{2}\right)=sinnx\cos\frac{x}{2}+\cos nx\sin\frac{x}{2}-\sin\frac{x}{2}$.

Obtinem $\sum_{k=1}^{n}\cos kx=\frac{1}{2}\left(\sin nxctg\frac{x}{2}+\cos nx-1\right), (\forall)n\in\mathbf{N}^*, (\forall)x\in\mathbf{R}\setminus 2\pi\mathbf{Z};$

e) $\int_0^{\pi}h(x)\cos nxdxx=\int_0^{\pi}h(x)\left(\frac{\sin nx}{n}\right)'dx=\frac{1}{n}\int_0^{\pi}h'(x)\sin nxdx.$

Avem

$\left|\frac{1}{n}\int_0^{\pi}h'(x)\sin nxdx\right|\le\frac{1}{n}\int_0^{\pi}|h'(x)|dx\le\frac{1}{n}\int_0^{\pi}Mdx=\frac{M\pi}{n}, (\forall)n\in\mathbf{N}^*$, unde $M=\sup_{x\in[0,\pi]}|h'(x)|$.

Obtinem $\lim_{n\to\infty}\int_0^{\pi}h(x)\cos nxdx=0$. Analog se arata ca $\lim_{n\to\infty}\int_0^{\pi}h(x)\sin nx=0$;

f) Avem $\lim_{x\to 0}\frac{g(x)-g(0)}{x}=\lim_{x\to 0}\frac{\left(\frac{1}{2\pi}x^2-x\right)\cos\frac{x}{2}+2\sin\frac{x}{2}}{x\sin\frac{x}{2}}=\frac{1}{\pi}$;

g) Din c) si d) $\Rightarrow b_n=\frac{1}{2}\left(\int_0^{\pi}g(x)\sin nxdx+\int_0^{\pi}f(x)\cos nxdx-\frac{1}{2}a_0\right)=$

$=\frac{1}{2}\left(\int_0^{\pi}g(x)\sin nxdx+\int_0^{\pi}f(x)\cos nxdx+\frac{\pi^2}{6}\right), (\forall)n\in\mathbf{N}^*$.

Utilizand f) si e) obtinem $\lim_{n\to\infty}b_n=\frac{\pi^2}{6}$.